Paracompact Spaces

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June 1, 2020

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1 Paracompact Spaces

While local compactness has appeared frequently in our work it can be an awkward property to work with. It neither generalises compactness nor guarantees that any of the good local properties are observed at larger scales. For this reason it is often desirable to find other ways to capture the good properties enjoyed by compact spaces that make them more apparent on a global level. It is for reason that we now introduce the notion of *paracompactness*. This is something that does generalise compactness, and it has many similarities with metric topology. It turns out that paracompactness is especially useful when trying to globalise observed local properties.

A key feature of paracompactness is the existence of suitable *partitions of unity*. These are collections of real-valued functions which can be used for gluing together maps and homotopies. For our intents, their existence is a suitable way to characterise paracompactness. On the other hand it is often more important to simply have some partition of unity, rather than make blanket assumptions on their existence. Thus both objects will be of independent interest to us.

Definition 1 *A collection $\mathcal{S}=\{S_i\}_{i\in\mathcal{I}}$ of subsets $S_i\subseteq X$ of a space X, is said to **cover** X if $\bigcup_{i\in\mathcal{I}}S_i=X$. A cover $\mathcal{S}$ is said to be **open (closed)** if each S_i is open (closed). If $\mathcal{S}'=\{S'_j\}_{j\in\mathcal{J}}$ is a second cover, then we say that;*

*1) $\mathcal{S}'$ is a **subcover** of $\mathcal{S}$ if $\mathcal{J}\subseteq\mathcal{I}$ and $S'_j=S_j$ for each $j\in\mathcal{J}$.*

*2) $\mathcal{S}'$ is a **refinement** of $\mathcal{S}$ if for each $S'_j\in\mathcal{S}'$ there exists $S_i\in\mathcal{S}$ with $S'_j\subseteq S_i$. If $\mathcal{I}=\mathcal{J}$, then we say that the refinement $\mathcal{S}'$ of $\mathcal{S}$ is **precise**.*

*3) $\mathcal{S}'$ is a **shrinking** of $\mathcal{S}$ if $\mathcal{J}=\mathcal{I}$ and for each $S'_j\in\mathcal{S}'$ it holds that $\overline{S}'_j\subseteq S_j$.* $\square$

Note that a subcover is a refinement, and a shrinking is a refinement. A space is compact if and only if every open cover has a finite subcover if and only if every open cover has a finite refinement.

Definition 2 *A collection of subsets $\mathcal{S} = \{S_i\}_{i\in\mathcal{I}}$ of a space X is said to be;*

1) ***discrete*** *if each point $x \in X$ has a neighbourhood intersecting at most one of the sets in $\mathcal{S}$.*

2) ***locally-finite*** *if each point $x \in X$ has a neighbourhood intersecting at most finitely many of the sets in $\mathcal{S}$.*

3) ***point-finite*** *if each point $x \in X$ is contained in at most finitely many of the sets in $\mathcal{S}$.* $\square$

Clearly a discrete or finite family of subsets is locally-finite, and a locally-finite family is point-finite. A subfamily of a discrete/locally-finite/point-finite family is itself discrete/locally-finite/point-finite.

Lemma 1.1 *Let X be a space and $\mathcal{S} = \{S_i\}_\mathcal{I}$ a family of subsets $S_i \subseteq X$. Then the following statements hold.*

1) *If $\mathcal{S}$ is locally-finite, then $\overline{\bigcup_\mathcal{I} S_i} = \bigcup_\mathcal{I} \overline{S}_i$.*

2) *If $\mathcal{S}$ is locally-finite and each S_i is closed, then $\bigcup_\mathcal{I} S_i$ is closed. If each S_i is both open and closed, then $\bigcup_\mathcal{I} S_i$ is both open and closed.*

3) *If $\mathcal{S}$ is locally-finite (discrete), then the family $\{\overline{S}_i\}_\mathcal{I}$ is also locally-finite (discrete).*

4) *If $\mathcal{S}$ is locally-finite and $K \subseteq X$ is compact, then K meets at most finitely many of the sets in $\mathcal{S}$.*

Proof 1) Clearly $\bigcup_\mathcal{I} \overline{S}_i \subseteq \overline{\bigcup_\mathcal{I} S_i}$ since each $\overline{S}_i \subseteq \overline{\bigcup_\mathcal{I} S_i}$. To show the reverse inclusion let $x \in \overline{\bigcup_\mathcal{I} S_i}$ and choose a neighbourhood V of x meeting only finitely many of the S_i. Then given an arbitrary neighbourhood U of x, the set $U \cap V$ meets $\bigcup_\mathcal{I} S_i$ nontrivially by assumption, but by construction meets only finitely many of the S_i nontrivially, say $S_1, \ldots, S_n$. This implies that U meets $\bigcup_{i=1}^n S_i$, and since U was arbitrary we can conclude that $x \in \overline{\bigcup_{i=1}^n S_i}$. This now implies that

$$x \in \overline{\bigcup_{i=1}^n S_i} = \bigcup_{i=1}^n \overline{S_i} \subseteq \bigcup_\mathcal{I} \overline{S}_i. \tag{1.1}$$

Parts 2) and 3) now follow easily.

For part 4) we can cover K with a finite number of open sets, each of which meets at most finitely many of the sets in $\mathcal{S}$. $\blacksquare$

Lemma 1.2 *Let $\mathcal{U}$ be an open covering of a space X.*

1) *If $\mathcal{U}$ admits a point-finite open refinement, then it admits a precise point-finite open refinement.*

2) If $\mathcal{U}$ admits a locally-finite open (closed) refinement, then it admits a precise locally-finite open (closed) refinement.

Proof We prove only the second statement since the first is similar. Choose a locally-finite open (closed) refinement $\mathcal{V}$ of $\mathcal{U}$. For each $V \in \mathcal{V}$ choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then for each $U \in \mathcal{U}$ set

$$W_U = \bigcup_{U_V = U} V. \tag{1.2}$$

We claim that $\mathcal{W} = \{W_U\}_{U \in \mathcal{U}}$ is a precise locally-finite refinement of $\mathcal{U}$. Moreover it is open (closed) when so is $\mathcal{V}$.

Now $\mathcal{W}$ is certainly a covering since each $V \subseteq W_{V_U}$, and it refines $\mathcal{U}$ precisely since $W_U \subseteq U$. If each V is open, then so is each W_U. On the other hand, if each V is closed, then Lemma 1.1 applies to the locally-finite family $\mathcal{V}$, and shows that so is each W_U.

To see that $\mathcal{W}$ is locally-finite let $x \in X$ and choose a neighbourhood Q of x which meets only finitely many members of $\mathcal{V}$. Then if

$$Q \cap W_U = \bigcup_{U_V = U} Q \cap V \neq \emptyset \tag{1.3}$$

we have $Q \cap V \neq \emptyset$ for some $V \in \mathcal{V}$ with $U_V = U$. But by assumption this can only happen for finitely many such V. Hence $Q \cap W_U$ can only be nonempty for finitely many $W_U \in \mathcal{W}$. We conclude from this that $\mathcal{W}$ is locally-finite. ∎

We now formulate the definition of paracompactness. Following Bourbaki [1] and Engelking [2] we include the Hausdorff assumption. James's definition [5] replaces this with regularity (compare 1.4 below). Notable authors such as Nagata [7] and Munkres [6] do not include any separation assumptions.

Definition 3 *A Hausdorff space X is said to be* ***paracompact*** *if any open cover of it has a locally-finite open refinement.* □

Clearly any discrete space is paracompact. With Munkres's definition an indiscrete space is paracompact, but for us it fails to be so. Since a finite subcover is a locally-finite refinement, we see that a compact Hausdorff space is paracompact. Thus the definition is indeed a suitable generalisation of compactness. It's easy to see, however, that the definition does capture something new.

Example 1.1 *The space $\mathbb{R}^n$ is paracompact. For given an open covering $\mathcal{U}$ of $\mathbb{R}^n$ set $B_0 = \emptyset$ and for each $n \in \mathbb{N}$ let B_n be the open ball of radius n centred at the origin. Then since the closure of B_n is compact there is a finite subfamily $\mathcal{U}(n) \subseteq \mathcal{U}$ which covers B_n. Set*

$$\mathcal{V}(n) = \{U \cap (\mathbb{R}^n \setminus \overline{B}_n) \mid U \in \mathcal{U}(n)\}. \tag{1.4}$$

Then each $\mathcal{V}(n)$ is finite and the family

$$\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}(n) \tag{1.5}$$

refines $\mathcal{U}$. Moreover $\mathcal{V}$ is locally-finite, since each B_n intersects only finitely many of its members. To see finally that $\mathcal{V}$ is a covering let $x \in \mathbb{R}^n$ and k the smallest integer such that $x \in \overline{B}_k$. Then clearly x belongs to some member of $\mathcal{V}(k)$. □

Proposition 1.3 *A closed subspace of a paracompact space X is itself paracompact.*

Proof Assume that $\mathcal{U}$ is an open cover of a closed subset $C \subseteq X$. Extend $\mathcal{U}$ to a covering of C by open sets in X and let $\mathcal{U}' = \mathcal{U} \cup \{X \setminus C\}$. Then $\mathcal{U}'$ has a locally-finite open refinement, which also refines $\mathcal{U}$. ∎

Note that in general arbitrary, and even open, subspaces of a paracompact space X may fail to be paracompact. One can show, however, that arbitrary subspaces of X are paracompact if and only if open subspaces of X are paracompact [1].

Proposition 1.4 *A paracompact space X is normal.*

Proof We first show that X is regular. Thus we must find separating neighbourhoods for a given closed subset $A \subseteq X$ and any point $x \in X \setminus A$. To begin use the Hausdorff assumption on X, to find, for each point $y \in A$, disjoint open neighbourhoods U_y of x and V_y of y. Now $A \subseteq X$ is closed, and hence paracompact (cf. Pr. 1.3). Thus we can find a locally-finite family $\mathcal{V}$ of open subsets of X which covers A and refines the family $\{V_y\}_{y \in A}$. Let $V \subseteq X$ be the union of all the sets in $\mathcal{V}$. Then $A \subseteq V$ and $x \notin V$. Moreover, according to Lemma 1.1, the closure $\overline{V}$ is the union of the closures of the sets in $\mathcal{V}$. This implies that $x \notin \overline{V}$, since for any given element of $\mathcal{V}$ we can find a disjoint neighbourhood of x. Thus V and $X \setminus \overline{V}$ are the required separating neighbourhoods, and X is regular.

To show now that X is normal assume that $B \subseteq X$ is a second closed set disjoint from A. Using the knowledge now that X is regular we can improve the argument above by replacing the sets V_y, $y \in A$, with open sets whose closures are disjoint from B. The remainder of the argument goes through to construct disjoint neighbourhoods V of A and $X \setminus \overline{V}$ of B. ∎

In fact a paracompact space satisfies stronger normality conditions. In particular Stone's Theorem states that a Hausdorff space is paracompact if and only if it is *fully normal* [7]. This is difficult to prove and we shall not attempt it. A direct consequence, however, is the following important statement.

Theorem 1.5 (Stone) *Every metrisable space is paracompact.* ∎

Thus we greatly generalise Example 1.1, and so find plentiful examples of paracompact spaces which are not compact.

Corollary 1.6 *Every smooth manifold is paracompact.*

Proof This is a consequence of 1.5 and Urysohn's Metrisation Theorem [6] § 34 pg. 215 which states more generally that any regular second-countable space is metrisable. ∎

Remark In our discussion of partitions of unity we indicate how this corollary leads to the conclusion that every compact manifold embeds in some euclidean space. □

Corollary 1.7 *Every locally-finite CW complex is paracompact.*

Proof This is a consequence of 1.5 and Theorem 1.5.17 in [3], which states that a CW complex is locally-finite if and only if it is metrisable if and only if it is first-countable. ∎

Remark The cited result in [3] is actually a consequence of their embedding Theorem 1.5.16, which states that a countable, locally-finite CW complex embeds in the Hilbert cube, and can be embedding in $\mathbb{R}^{2n+1}$ if it has dimension $\leq n$. While not every CW complex is metrisable, it is in fact true that every CW complex is paracompact, and we shall indicate a proof of this in a later lecture when we discuss CW complexes in detail. $\square$

Paracompactness is not in general passed to quotient spaces or continuous images. However the following result due to E. Michael is frequently useful.

Theorem 1.8 *Assume that $f : X \to Y$ is a closed surjection of a paracompact space X onto a space Y. Then Y is paracompact.* $\blacksquare$

Note that this even implies that Y is Hausdorff. A proof of the theorem can be found in [2]. We, however, need a result in the opposite direction.

Proposition 1.9 *Assume that $f : X \to Y$ is a proper surjection of a Hausdorff space X onto a paracompact space Y. Then X is paracompact.*

Proof Assume given an open cover $\mathcal{U} = \{U_i\}_{\mathcal{I}}$ of X. For each $y \in Y$ the inverse image $f^{-1}(y)$ is compact, so can be covered by the members of a finite subfamily $\mathcal{U}(Y) \subseteq \mathcal{U}$. Moreover, since f is closed, there is an open $V(y) \subseteq Y$ such that

$$f^{-1}(y) \subseteq f^{-1}(V(y)) \subseteq \bigcup_{\mathcal{U}(y)} U. \tag{1.6}$$

Now the collection $\{V(y)\}$ covers Y, so has a locally-finite open refinement $\mathcal{W}$. This implies that the inverse images $f^{-1}\mathcal{W} = \{f^{-1}(W) \mid W \in \mathcal{W}\}$ form a locally-finite open cover of X. For each $W \in \mathcal{W}$ we find $y_W \in Y$ such that $f^{-1}(W) \subseteq f^{-1}(V(y))$. Then

$$\{f^{-1}(W) \cap U \mid W \in \mathcal{W},\ U \in \mathcal{U}(y_W)\} \tag{1.7}$$

is a locally-finite family refining $\mathcal{U}$. $\blacksquare$

Clearly any disjoint union of paracompact spaces is paracompact. On the other hand, since normality is not productive, one might guess that neither is paracompactness, and indeed this is true.

Example 1.2 *The* ***Sorgenfrey Line*** *$\mathcal{L}$ is the real line given the topology generated by the basis of half open intervals $(a, b]$ where $a < b$. Then $\mathcal{L}$ is normal Hausdorff, but it is known that the square $\mathcal{L} \times \mathcal{L}$ is not normal. More relevantly, $\mathcal{L}$ is Lindelöf. That is, any open cover of it has a countable subcover [2] pg. 194. Clearly this implies that $\mathcal{L}$ is paracompact. However since $\mathcal{L} \times \mathcal{L}$ is not normal, this product cannot be paracompact (cf. Pr. 1.4).* $\square$

Let us record these observations formally.

Proposition 1.10 *The following statements hold.*

1) A disjoint union of spaces is paracompact if and only if each summand is paracompact.

2) A product of (even finitely many) paracompact spaces may fail to be paracompact.

3) A product of a paracompact space and a compact Hausdorff space is paracompact.

Proof Only the last item needs to be addressed and this is an easy consequence of Proposition 1.9: if X is Haudorff and K is compact, then the projection $X \times K \to X$ is proper. $\blacksquare$

We give next an example of an interesting yet pathological non-paracompact space. We make reference to this space, the *long ray*, in the lecture notes when we discuss an example of a locally trivial map which is not a fibration.

Example 1.3 *Let W be set of all countable ordinal numbers. For fixed $\alpha \in W$ define the half-open 'intervals'*

$$\alpha \uparrow = \{\beta \in W \mid \beta > \alpha\}, \qquad \alpha \downarrow \{\beta \in W \mid \beta < \alpha\}. \tag{1.8}$$

These sets form a subase for a topology on W as we vary $\alpha \in W$ across all points.

Next we define a space $\mathbb{L}$ which we call the ***long ray****. The idea is to glue in a copy of the unit interval between each ordinal in W and its succesor. If we replace W with the set of all finite ordinals, then the construction will yield the open interval $[0, \infty)$. Hence truly $\mathbb{L}$ will live up to its name. Explicitly we put*

$$\mathbb{L} = W \times I / \left[(\alpha, 1) \sim (\alpha + 1, 0)\right]. \tag{1.9}$$

We write the equivalence class of (α, s) as $\alpha + s$. Then with a small abuse of notation, we can write each element of $\mathbb{L}$ in the form $\lambda + t$, where λ is a limit ordinal and $t \in [0, \infty)$.

The long ray has many properties in common with the so-called long line*, which is formed by gluing together two copies of $\mathbb{L}$ at their origins. We don't need this space, but we will compile the following list of properties of the long ray.*

- *$\mathbb{L}$ is not compact. If for $\alpha \in W$ we write $U_\alpha = \{\beta + s \mid \beta \leq \alpha, s \in [0, 1)\}$, then $\{U_\alpha\}_{\alpha \in W}$ if an open cover of $\mathbb{L}$ with no finite subcover. In fact it has no countable subcover, so $\mathbb{L}$ is not even Lindelöf.*

- *$\mathbb{L}$ is locally Euclidean with boundary. It is easy to see that if α is a finite ordinal, then the set of points $\alpha + t$ is homeomorphic to $[0, \infty)$. It is a little harder to construct a euclidean neighbourhood in the case that α is infinite, but the idea is the same. See Gauld [4] for details.*

- *$\mathbb{L}$ is path-connected since any given ordinal is contained in a Euclidean neighbourhood containing the origin. In fact, using this idea its not difficult to see that there are no essential maps into $\mathbb{L}$ from a compact space. In particular all homotopy groups of $\mathbb{L}$ are trivial, as are all its singular homology groups.*

- *$\mathbb{L}$ is not contractible. For if $F : L \times I \to L$ is a contracting homotopy with $F_0 = id_L$, then each F_t is a self-map of L whose image is a (possibly unbounded) interval. If $A = \{t \in I \mid F_t(L) \text{ bounded}\}$, then $0 \notin A$ and $1 \in A$. However we can show that A is both closed and open in I, and so derive a contradiction to the existence of F.*

- $\mathbb{L}$ *is clearly completely regular Hausdorff.*

- $\mathbb{L}$ *is locally path-connected and locally compact since it is locally Euclidean.*

- $\mathbb{L}$ *is first-countable, but cannot be second-countable since it has an uncountable discrete subset.*

- $\mathbb{L}$ *is not paracompact, since this would necessarily imply that it was second-countable. The argument here is that a locally compact Hausdorff space X is σ-compact. i.e. can be expressed as a countable union of compact subspaces. In the case that X is locally Euclidean, parcompactenss can then be used to construct a cover X by countably many coordinate charts. The coordinate charts each carry a second-countable topology, and the covering by countably many induces one such on X. Since $\mathbb{L}$ is locally Euclidean, the same argument applies to it.*

Of note is the fact that the long ray $\mathbb{L}$ meets all the conditions for it to be a smooth manifold (with boundary) other than being second countable. If we define

$$\mathbb{L}_+ = \mathbb{L} \setminus \{(0,0)\} \tag{1.10}$$

then the same statements as above apply, except that this space is now locally Euclidean without boundary. □

We end this section with a result on pushouts.

Proposition 1.11 *Let X, Y be spaces, $A \subseteq X$ be a subspace, and $f : A \to B$ a continuous map. Denote by $B \cup_f X$ the adjunction space formed in the following pushout*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ B \arrow[r] & B \cup_f X. \end{tikzcd} \tag{1.11}$$

If both X, Y are paracompact and $A \subseteq X$ is closed, then $B \cup_f X$ is paracompact. In particular $B \cup_f X$ is Hausdorff. □

Proof By Proposition 1.10, the sum $B \sqcup X$ is paracompact. Moreover, since $A \subseteq X$ is closed, the canonical quotient $B \sqcup X \to B \cup_f X$ is a closed map. Thus we can apply Michael's result 1.8 to see that $B \cup_f X$ is paracompact. ∎

2 Paracompactness and Partitions of Unity

The goal of this section is to prove Theorem 2.3, which characterises paracompact spaces in terms of the numerability of their open covers. For this we need a pair of auxiliary results. The reader wishing to understand this section should consult the accompanying notes on partitions of unity.

Proposition 2.1 *Every open covering $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ of a paracompact space X has a locally-finite shrinking.*

Proof Let $\mathcal{V}'$ be the set of all open $V \subseteq X$ such that $V \subseteq U_i$ for some i. Since X is regular (cf. Pr. 1.4) the members of $\mathcal{V}'$ cover X, and hence $\mathcal{V}'$ is an open refinement of $\mathcal{U}$. Now choose a locally-finite open refinement $\mathcal{V}$ of $\mathcal{V}'$. Then $\mathcal{V}$ is a locally-finite open refinement of $\mathcal{U}$, and we check that $\overline{\mathcal{V}} = \{\overline{V} \mid V \in \mathcal{V}\}$ is a closed locally-finite refinement of $\mathcal{U}$.

Now apply Lemma 1.2 to convert $\mathcal{V}$ into a precise open refinement $\mathcal{W}$ of $\mathcal{U}$, and to convert $\overline{\mathcal{V}}$ into a precise closed refinement $\mathcal{W}'$ of $\mathcal{U}$. We need to check some details from the proof of the lemma, but when we do this we see that the closures of the elements of $\mathcal{W}$ are exactly the members of $\mathcal{W}'$. That is, $\mathcal{W}$ is a shrinking of $\mathcal{U}$. ∎

The following was proved in the lecture on partitions of unity, and we refer the reader there for its proof.

Theorem 2.2 *Let $\{\sigma_i\}_{i\in\mathcal{I}}$ be a partition of unity on a space X. Then there exists a locally-finite partition of unity $\{\xi_i\}_{i\in\mathcal{I}}$ on X indexed by the same set such that for each $i \in \mathcal{I}$ it holds that*

$$Supp(\xi_i) \subseteq \sigma_i^{-1}(0,1]. \tag{2.1}$$

In particular the open covering $\{\sigma_i^{-1}(0,1]\}_{\mathcal{I}}$ is numerable. ∎

Theorem 2.3 *Let X be a T_1-space. Then the following are equivalent.*

1) *X is paracompact.*

2) *Each open cover of X is numerable.*

3) *Each open cover of X has a partition of unity subordinated to it.*

Proof *i)* $\Rightarrow$ *ii)* Assume that X is paracompact and let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an open cover. Then by Proposition 2.1 we can find a locally-finite shrinking $\mathcal{V} = \{V_i\}_{i\in\mathcal{I}}$ of $\mathcal{U}$. Now use Urysohn's Lemma to find for each $i \in \mathcal{I}$ a continuous function $\pi_i : X \to I$ such that *i)* $\pi_i(x) = 0$ for $x \notin U_i$, and *ii)* $\pi_i(x) = 1$ for $x \in \overline{V}_i$. Since the family $\mathcal{V}$ is locally-finite the sum

$$\pi(x) = \sum_{i\in\mathcal{I}} \pi_i(x) \tag{2.2}$$

defines a continuous function $\pi : X \to [0,\infty)$. Also, since each $x \in X$ is contained in some $\overline{V}_i$, the function π is strictly positive. Now normalise the π_i by dividing by π to get the required partition of unity.

Now the implication *ii)* $\Rightarrow$ *iii)* is clear, so to complete we need only prove that *iii)* $\Rightarrow$ *i)*. Thus let $\mathcal{U}$ be an open covering of X with a subordinated partition of unity. Then Theorem 2.2 shows that $\mathcal{U}$ is numerable. Choosing any numeration for $\mathcal{U}$, the family of cozero sets is a locally-finite open refinement of $\mathcal{U}$.

Thus to complete we need only show that the T_1 space X is Hausdorff. So let $x, y \in X$ be distinct points and consider the open cover $\mathcal{U} = \{U_x, U_y\}$, where

$$U_x = X \setminus \{x\}, \qquad U_y = X \setminus \{y\}. \tag{2.3}$$

Choose a partition of unity $\{\xi_i\}_{\mathcal{J}}$ subordinated to $\mathcal{U}$ and find $j = j(x) \in \mathcal{J}$ such that $\xi_j(x) > 0$. Since $\xi_j^{-1}(0,1] \subseteq X \setminus \{y\}$ we have $\xi_j(y) = 0$, and this implies that

$$\xi_j^{-1}(0,1], \qquad \xi_j^{-1}[0, \xi_j(x)/2) \tag{2.4}$$

are a pair of disjoint open sets containing x, y respectively. $\blacksquare$

References

[1] N. Bourbaki *General Topology I*, Addison-Wesley, (1966).

[2] R. Engelking, *General Topology*, Heldermann Verlag, (1989)

[3] R. Fritsch, R. Piccinini, *Cellular Structures in Topology*, Cambridge University Press, (1990).

[4] D. Gauld, *Non-Metrisable Manifolds*, Springer, (2014).

[5] I. James, *General Topology and Homotopy Theory*, Springer-Verlag New York, (1984).

[6] J. Munkres, *Topology*, Second Edition, Prentice Hall, (2000).

[7] J. Nagata, *Modern General Topology*, Second Edition, North-Holland, (1985).